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A NOTE ON THE WEIGHTED COVARIANCE SET IN C^* - ALGEBRA

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ABSTRACT

In this article, we show that a well-known property of covariance set, namely the characterization of covariance set in terms of commutators, remains valid for the weighted covariance set in C^*-algebras.

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INTRODUCTION

The literature on generalized inverses is quite extensive. *Moore*[1,2] introduced and studied the general reciprocal during the years 1910-1920. The general reciprocal was find out again by Penrose[3] and is recently the Moore-Penrose inverse. The weighted Moore-Penrose inverse for matrices was introduced by Ben[4], Chipman[5], Khosravi[6], Meenakshi[7], Patricio[8], Prasad,[9] Robinson[10], Schwerdtfeger[11] *and* Sun[12]. Also it was studied in an indefinite inner product space by Kamaraj[13] . Wei[14] presented a connected representation theorem of the weighted MP-inverse in Hilbert spaces., Koliha[15] introduced weighed MP--inverses and obtained the reverse order rule for those in C^* -algebras.

In whole the article η will be a C^*-algebra. We say that $a \in \eta$ is *regular* if there is $a \in \eta$ such that

$$aba = a.$$

We say that $b \in \eta$ is a MP-inverse of if

$$aba = a,\ bab = b,\ (ab)^* = ab \text{ and } (ba)^* = ba.$$

It is fully known that every regular element in a C^* -algebra, has the Moore-Penrose inverse (We denote it briefly by MP-inverse). We shall mark denote the MP--inverse of a by $a^\dagger$.

In what follow, we use η^{-1} and $\eta^\dagger$ for the set of a invertible and MP- *invertible elements* of η, respectively. The commutator of *x, y* in η is denoted by *[x,y] =xy-yx*. Suppose that *e* and *f* are positive elements in η^{-1}. Let $a^\dagger$ be a MP-inverse of *a*. Set

$$y = f^{1/2} a^\dagger e^{-(1/2)} \text{ and } b = e^{1/2} a f^{-(1/2)},$$

then we get

$$yby = y, byb = b, \quad (yb)^* = f^{-1} ybf \text{ and } (by)^* = e^{-1} bye.$$

In this case, *y* is called the *weighted Moor-Penrose inverse* of *b* with weights *e* and *f*. Many noteworthy properties of weighted MP-inverse for matrices were introduced and studied by Rao[16] .

Assume that *a* is an element in η^{-1}. Then a^{-1} is *covariant* under η^{-1}, i.e., for each $b \in \eta^{-1}$ we have

$$\left(bab^{-1}\right)^{-1} = ba^{-1}b^{-1}.$$

Alizadeh[17] was shown that, the elements of $\eta^\dagger$ are not covariant under η^{-1}. For $a \in \eta^\dagger$ with MP-inverse $a^\dagger$, we define the *covariance set* by

$$\square\,(a) = \left\{ b \in \eta^{-1} : \left(bab^{-1}\right)^\dagger = ba^\dagger b^{-1} \right\}. \quad (1)$$

Here we generalize some results the *weighted covariance set* of a regular element in C^* -algebras.

Weighted Moore-Penrose inverses and covariance set

Let us start with the following definition.

Definition 1 Let η be a C^* -algebra and e, f be a be *two positive elements* in η^{-1}. *We say that an element* $a \in \eta$ has *a weighted MP-inverse with weights* e, f *if there exist* $b \in \eta$ *such that*

$$aba = a, bab = b, \quad (eab)^* = eab \quad \textit{and} \quad (baf)^* = baf. \quad (2)$$

If the weighted MP-inverse with weights e, f exists, then it is unique. We will denote it by $a^\dagger_{e,f}$.

Each regular element in a C^* -algebra has a weighted MP-inverse [9, Theorem 4] and it can be written as

$$a^\dagger_{e,f} = f^{-\frac{1}{2}} \left(e^{\frac{1}{2}} a f^{-\frac{1}{2}} \right)^\dagger e^{\frac{1}{2}}. \quad (3)$$

Remark 2 *Suppose that* a **is** *a regular element in* η.

(i) *If* $e = f = 1$. *then the MP-inverse coincide, i.e.,*

$$a_{e,f}^{\dagger} = a_{1,1}^{\dagger} = a^{\dagger}$$

(ii) *If* $e = 1$ *or* $f = 1$, *then*

$$a_{1,f}^{\dagger} = f^{-\frac{1}{2}\left(af^{-\frac{1}{2}}\right)^{\dagger}} \quad and \quad a_{e,1}^{\dagger} = \left(e^{\frac{1}{2}}a\right)^{\dagger} e^{\frac{1}{2}} \tag{4}$$

(iii) According to the Theorem 2.3 studied by Alizadeh[17] *, one can easily check that*

$$aa_{e,1}^{\dagger} = aa^{\dagger} \quad \text{and} \quad a_{e,1}^{\dagger}a = a^{\dagger}a. \tag{5}$$

(iv) Let a *be positive element in* $\eta^{\dagger}$ *and* $\varphi : \eta \to \mu$ *be a homomorphism. Then* $\varphi\left(a^{\dagger}\right)$ *is positive in* μ *because*

$$\varphi\left(a^{\dagger}\right) = \varphi\left(a^{\dagger}aa^{\dagger}\right) = \varphi\left(\left(a^{\frac{1}{2}}a^{\dagger}\right)^{*} a^{\frac{1}{2}}a^{\dagger}\right) = \varphi\left(a^{\frac{1}{2}}a^{\dagger}\right).$$

Also by using of (3), we get if $a \in \eta^{\dagger}$ is positive, then $a_{e,f}^{\dagger}$ and $\varphi(a_{e,f}^{\dagger})$ are positive in η and μ respectively.

Let e and f be positive elements in η^{-1}.

Now, we can introduce the concept of MP--inverse in η_e The MP- inverse of $a \in \eta_e$ is $b \in \eta_e$ such that

$$aba = a,\ bab = b, \quad (ab)^{*e} = ab \qquad \text{and} \quad (ba)^{*f} = ba.$$

We shall write $a^{\updownarrow}$ for the MP-inverse of $a \in \eta_e$.

The next proposition shows that (2) and (6) are equivalent.

Proposition 3 Assume that a is a regular element in η and e, f are positive in η^{-1}. Then $a^{\updownarrow} = a_{e,f}^{\dagger}$.

Proof. Since

$$(eab)^{*} = eab \Leftrightarrow e^{-1}(eab)^{*} = ab \Leftrightarrow e^{-1}(ab)^{*}e = ab \Leftrightarrow (ab)^{*e} = ab$$

and similarly

$$(abf)^{*} = abf \Leftrightarrow (ba)^{*f} = ba$$

The uniqueness of $a_{e,f}^{\dagger}$ implies that $a^{\updownarrow} = a_{e,f}^{\dagger}$.

From the above proposition we conclude that each regular elementhas a MP-inverse in η_e. In fact, from (3) we get $a^{\updownarrow} = f^{-\frac{1}{2}}\left(e^{\frac{1}{2}}af^{-\frac{1}{2}}\right)^{\dagger} e^{\frac{1}{2}}$. Note that since $a^{\updownarrow}$ is unique, thus $\left(a^{\updownarrow}\right)^{*e} = \left(a^{*e}\right)^{\updownarrow}$. For weighted MP-inverses, we associate the *weighted involution* in η .Assume that e, f are positive elements in η^{-1}. Define $x^{*e,f} = e^{-1}x^{*}f$. It is easy to see that the mapping $x \to x^{*e,f}$ is an involution on η which satisfies $\left(x^{*e,f}\right)^{*} = \left(x^{*}\right)^{*e,f}$. Note that $\left(x^{*e,f}\right)^{\dagger} = \left(x^{\dagger}\right)^{*e,f}$ need not hold for this involution. The notion of covariance set is now extended to the weighted MP-inverses.

Suppose that $a \in \eta$ is regular with MP-inverse $a^{\dagger}$ and e, f are positive elements in η^{-1}. We define the

Weighted covariance set by

$\square_{e,f}(a) = \{\, b \in \eta^{-1} : b^{-1}a^{\dagger}b$ is weighted MP-inverse of $b^{-1}ab$ with weights e, f }.

Theorem *4 Assume that $a \in \eta^{\dagger}$ with MP-inverse $a^{\dagger}$ and e, f are positive elements in η^{-1}. Then the following statements are equivalent*:

(i) $b \in \square_{e,f}(a)$;

(ii) $\left[a^{\dagger}a, b^{*}b\right] = 0$ and $\left[aa^{\dagger}, b^{*}fb\right] = 0$.

Proof. $(i) \Rightarrow (ii)$ Suppose that $b \in \square_{e,f}(a)$. then $b^{-1}a^{\dagger}b$ is the weighted MP-inverse of $b^{-1}ab$ with weights e, f .Thus $\left(ba^{\dagger}ab^{-1}\right)^{*e} = ba^{\dagger}ab^{-1}$ and so $e^{-1}\left(ba^{\dagger}ab^{-1}\right)^{*} e = ba^{\dagger}ab^{-1}$. Thus

$\left(b^{*}e\right)^{-1} a^{\dagger}ab^{*}eb = ba^{\dagger}a$. This implies that $\left[a^{\dagger}a, b^{*}eb\right] = 0$. Similarly the equality $\left(ba^{\dagger}ab^{-1}\right)^{*f} = ba^{\dagger}ab^{-1}$ implies $\left[a^{\dagger}a, b^{*}fb\right] = 0$.

$(ii) \Rightarrow (i)$ Since $a^{\dagger}$ is the MP-inverse of a, it suffices to show that $\left(ba^{\dagger}ab^{-1}\right)^{*e} = ba^{\dagger}ab^{-1}$ and $\left(baa^{\dagger}b^{-1}\right)^{*f} = ba^{\dagger}ab^{-1}$.By assumptions $\left[a^{\dagger}a, b^{*}eb\right] = 0$. From this we get $e^{-1}\left(b^{*}\right)^{-1} a^{\dagger}ab^{*}eb = ba^{\dagger}a$. Thus $e^{-1}\left(ba^{\dagger}ab^{-1}\right)^{*} e = ba^{\dagger}ab^{-1}$ and so $\left(ba^{\dagger}ab^{-1}\right)^{*e} = ba^{\dagger}ab^{-1}$. In a similar manner we obtain $\left(ba^{\dagger}ab^{-1}\right)^{*f} = ba^{\dagger}ab^{-1}$.

Corollary 5 *Assume that $a \in \eta^{\dagger}$ with MP-inverse $a^{\dagger}$ .Then*

$$\square_{e,f}(a) = \square_{f,e}\left(a^{\dagger}\right) = \square_{f,e}\left(a^{*}\right) = \square_{e,f}\left(aa^{\dagger}\right) \cap \square_{e,f}\left(a^{\dagger}a\right).$$

Proof. *An easy consequence of Theorem 4and Theorem 3.3 studied by* Alizadeh [17] .

Corollary 6 *Suppose that a and b are regular and simply polar elements with the same rang e ideals. Then $\square_{e,f}(a) = \square_{e,f}(b)$.*

Remark 7 *Assume that $a \in \Re^{\dagger}$ with MP-inverses $a^{\dagger}$.*

(i) *If* $e = f = 1$, *then the covariance set and the weighted covariance set of* a *coincide, that is* $\Box_{e,f}(a) = \Box(a)$;

(ii) $b \in \Box_{1,f}(a)$ *iff* $[a^{\dagger}a, b^{*}b] = 0$ *and* $[aa^{\dagger}, b^{*}fb] = 0$;

(iii) $b \in \Box_{e,1}(a)$ *iff* $[a^{\dagger}a, b^{*}eb] = 0$ *and* $[aa^{\dagger}, b^{*}b] = 0$;

(iv) *According to (5)and Corollary 5 we get*

$$\Box_{f,e}(a) = \Box_{e,f}(aa^{\dagger}_{1,f}) \cap \Box_{e,f}(a^{\dagger}_{e,1}a)$$

(v) *If* a *is simply polar, then*

$$\Box_{f,e}(a) = \Box_{e,f}(aa^{\dagger}_{1,f}) = \Box_{e,f}(a^{\dagger}_{e,1}a)$$

Proposition 8 *Let* $a, b \in \eta^{\dagger}$ *with MP-inverses* $a^{\dagger}$ *and* $b^{\dagger}$ *respectively. If* $a^{\dagger}b = 0 = ab^{\dagger}$ *and* $ba^{\dagger} = 0 = b^{\dagger}a$ *then* $\Box_{f,e}(a) \cap \Box_{e,f}(b) \subset \Box_{f,e}(a+b)$.

Proof. The proof is straightforward.

CONCLUSION

It is well known that the generalized inverses are very important both in theoretical and practical point of view. In this note we study the More-Penrose inverses in C*-algebras. If take he C*-algebra of matrices with the transpose conjugate, then it has a lot of applications in linear algebra.

REFERENCES

1. Moore E.H. , *On the reciprocal of the general algebraic matrix*, Bull. Amer. Math. Soc. **26** 394-395(1920)

2. Moore E.H. and Barnard R.W., General Analysis, Memories of the American Philosophical Society, I. American Philosophical Society, Philadelphia, Pennsylvania, Part 1, 197-209 (1935)

3. Penrose R. , *A generalized inverse for matrices*. Proc. Cambridge Philos. Soc.**51**, 406-413 (1955)

4. Ben-Israel A. and Greville T. N. E. , *Generalized Inverses*: Theory and Applications}, 2nd Edn. Springer-Verlag, New York, (2003)

5. Chipman J. S. , *On least squares with insufficient observation*, J. Amer. Statist. Assoc., 1078--111154 (1964)

6. Khosravi A. and Alizadeh M.H. *Generalized inverses of products*, Int. J. Appl. Math. **10** ,141-148(2002)

7. Meenakshi A. R. and Chinadurai V., *Some remarks on the covariance of the Moore-Penrose inverse,* Houtson J. Math. **18** , 167-174 (1992)

8. Patricio P. and Araujo C. M., *Moore-Penrose invertibility in involutory rings: the case* $aa^{\dagger} = bb^{\dagger}$, Linear and Multilinear Algebra, **58**, 445-452 (2010)

9. Prasad K. M., and Bapat R. B., *The generalized Moore-Penrose inverse,* Linear Algebra Appl. **165**, 59-69 (1992)

10. Robinson D. W., *On the covariance of the Moore-Penrose Invers* , Linear Algebra Appl. **61**, 91-99 (1984)

11. Schwerdtfeger H. , *On the covariance of Moore-Penrose inverse*, Linear Algebra Appl. **52/53**, 629-643 (1983)

12. Sun W. and Wei Y. , *Inverse order rule for weighted generalized inverse*, SIAM J. Matrix Anal. Appl. **19**, 772-775 (1998)

13. Kamaraj K. and Sivakumar K. C. ,*Moore-Penrose inverse in an indefinite inner product space,* J. App. Math. & Computing **19** ,297-310(2005)

14.Wei Y. , *The representation and approximation for the weighted Moore-Penrose inverse in Hilbert space,* Applied Mathematics and Computation **136** 47-486 (2003)

15. Koliha J.J., Djordjevic D. and Cvetkovic D., *Moore-Penrose inverse in rings with involution*, Linear Algebra Appl. **426** 371-381(2007)

16. Rao C.R. and Mitra S. K., *Generalized inverse of matrices and its applications*, Wiley and Sons, New York, (1971)

17. Alizadeh M.H. , *On the covariance of generalized inverse in* C^* -algebra, J. Numer. Anal. Indust. Appl. Math (JNAIAM). **5**, 135-139 (2011).